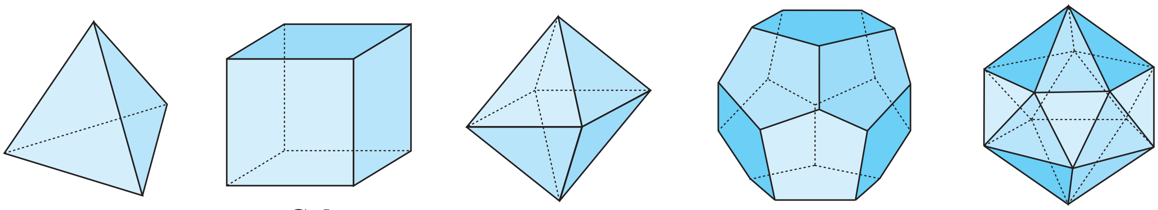
INVESTIGATION — PLATONIC SOLIDS

Platonic solids are regular polyhedra whose faces are all the same shape.

There are exactly five platonic solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron.



Tetrahedron   Cube   Octahedron   Dodecahedron   Icosahedron

Their existence has been known since the time of the ancient Greek civilisation.

In this investigation, prove that there are only five platonic solids using Euler's formula and the fact that all platonic solids are planar.

Suppose that the regular polyhedron P under consideration has v corners (vertices), e edges, and f faces (regions).

Since P is regular, the degree of each vertex is the same so we let the degrees be p where $p \geqslant 3$. Also, each region of the graph has the same shape, so we let the number of sides in each region be q, where $q \geqslant 3$.

Then $pv = qf = 2e$.

Also, since the solids are planar, we have Euler's relation $v - e + f = 2$.

1 Show that $8 = v(4-p) + f(4-q)$.

Since v and f are both positive, $(4-p)$ and $(4-q)$ cannot both be negative.

$\therefore$ either $4 - p \geqslant 0$ or $4 - q \geqslant 0$

$\therefore$ $3 \leqslant p \leqslant 4$ or $3 \leqslant q \leqslant 4$, though not necessarily both together.

We now have four cases to investigate: $p = 3$, $p = 4$, $q = 3$ and $q = 4$ and we must consider each of these in turn to complete our proof.

2 For the case $p = 3$, use $qf = 3v = 2e$ and the modified Euler relation derived in **1** to show that $f(6-q) = 12$, $f, q \in \mathbb{Z}^+$.

Using the factors of 12, we can consider the separate cases from the equation derived in **2**:

- $f = 1$ $\therefore$ $q < 0$ which is invalid
- $f = 2$ $\therefore$ $q = 0$ which is invalid {a region without edges}
- $f = 3$ $\therefore$ $q = 2 \Rightarrow v = 2$ which is invalid {3 vertices required for a region}
- $f = 4$ $\therefore$ $q = 3 \Rightarrow v = 4$, $e = 6$ a tetrahedron
- $f = 6$ $\therefore$ $q = 4 \Rightarrow v = 8$, $e = 12$ a cube
- $f = 12$ $\therefore$ $q = 5 \Rightarrow v = 20$, $e = 30$ a dodecahedron

3 For the case $p = 4$, use $qf = 4v = 2e$ and the modified Euler relation to show that $f(4-q) = 8$, $f, q \in \mathbb{Z}^+$. Use the procedure above and hence show that an octahedron is a platonic solid.

4 Repeat **3** for the cases $q = 3$ and $q = 4$.

Having considered all cases, you should have proven there are exactly five platonic solids.

Questions:

1 Draw a Schlegel diagram for each platonic solid.

2 Draw a Hamiltonian path on each diagram drawn in **1**.