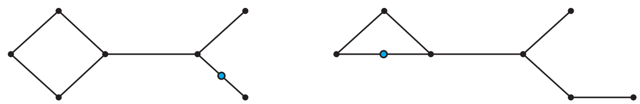
HOMEOMORPHIC GRAPHS AND THE THEOREM OF KURATOWSKI

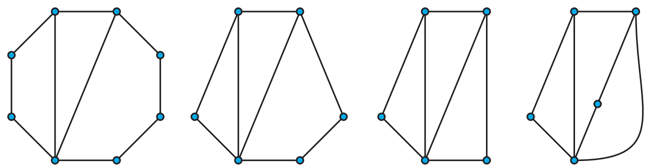
The Polish mathematician **Kazimierz Kuratowski** discovered an interesting property of planar and non-planar graphs. In fact, it is in honour of Kuratowski that the complete graphs are labelled K_n and the complete bipartite graphs are labelled $K_{m,n}$. However, before we can discuss Kuratowski's work, we need to look at **homeomorphism**.

Two graphs G and H are **homeomorphic** if they can be made isomorphic by inserting new vertices (of degree 2) into their existing edges.

For example, the two graphs below are homeomorphic since the addition of the coloured vertices shown will make them isomorphic.



Similarly, all of these graphs are homeomorphic:

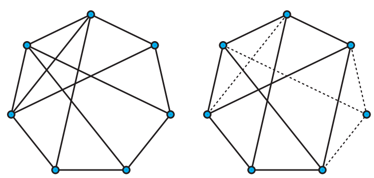


Now K_5 and $K_{3,3}$ are both non-planar. It is therefore clear that if either K_5 or $K_{3,3}$ is a subgraph of a graph G, then G must also be non-planar. However, Kuratowski extended this to say that every non-planar graph has a subgraph that is *homeomorphic* to either K_5 or $K_{3,3}$.

Formally we can state Kuratowski's theorem:

A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

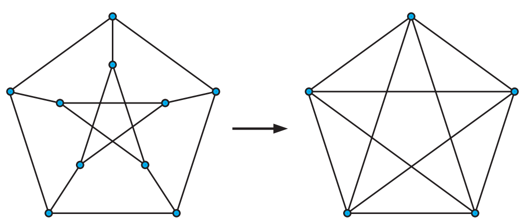
For example, these diagrams show a non-planar graph and a subgraph homeomorphic to $K_{3,3}$.



A further result is that:

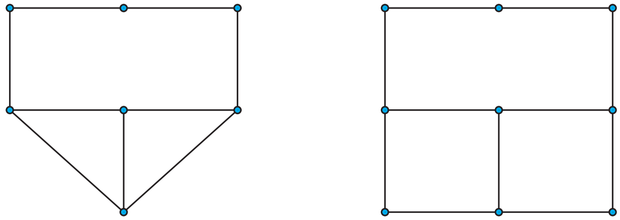
A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$ by removing edges from the subgraph and merging the adjacent vertices into one.

For example, we can contract the Peterson graph as shown below, thus proving the Peterson graph is non-planar.



EXERCISE

1 Show that these graphs are homeomorphic:



2 Show that all circuit graphs are homeomorphic to C_3.

3 Show that K_3 is homeomorphic to $K_{2,2}$.

4 Suppose G_1 has v_1 vertices and e_1 edges and that G_2 has v_2 vertices and e_2 edges and that G_1 is homeomorphic to G_2. Show that $e_1 - v_1 = e_2 - v_2$.

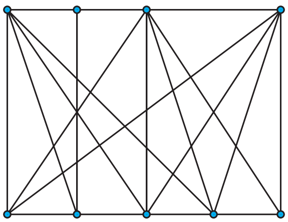
5 If G is Eulerian and H is homeomorphic to G, is H Eulerian?

6 If G is Hamiltonian and H is homeomorphic to G, is H Hamiltonian?

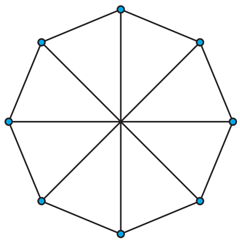
7 Use Kuratowski's theorem to show that K_n is non-planar for $n \geqslant 5$.

8 Use Kuratowski's theorem to show that the graphs below are non-planar.

a

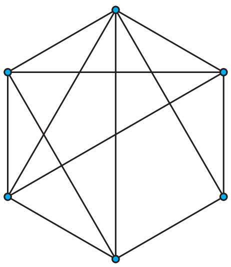


b



9 Can you use Kuratowski's theorem to show that the graphs below are non-planar?

a



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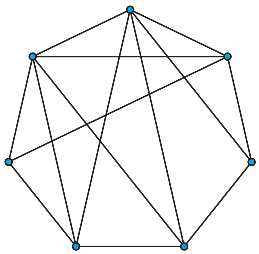
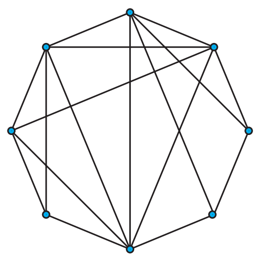
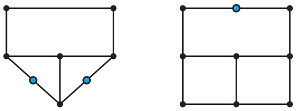


c



HOMEOMORPHIC GRAPHS AND THE THEOREM OF KURATOWSKI - ANSWERS

1 If we add the vertices shown, the resulting graphs are isomorphic.



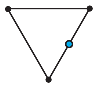
2 Consider the general circuit graph with n vertices, i.e., C_n where $n \geqslant 3$.

If we add a vertex of degree 2 into any existing edge, we generate the circuit graph with $n+1$ vertices, i.e., C_{n+1}.

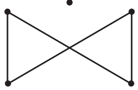
$\therefore$ C_n is homeomorphic to C_{n+1} for all $n \geqslant 3$.

$\therefore$ by induction, all circuit graphs are homeomorphic to C_3.

3 Given K_3, we can add the vertex shown:



The graph is now isomorphic to $K_{2,2}$:



Hence K_3 and $K_{2,2}$ are homeomorphic.

4 If G_1 and G_2 are homeomorphic, then we can add vertices of degree 2 into their existing edges in some manner so as to form isomorphic graphs H_1 and H_2.

We suppose H_1 and H_2 each have v vertices.

So, to form H_1 from G_1, we add $(v - v_1)$ vertices of degree 2. The sum of the degrees of the vertices of G_1 is $2e_1$, so the sum of the degrees of the vertices of H_1 is $2e_1 + 2\,(v - v_1)$.

Similarly, the sum of the degrees of the vertices of H_2 is $2e_2 + 2\,(v - v_2)$.

But H_1 and H_2 are homeomorphic, so

$$2e_1 + 2\,(v - v_1) = 2e_2 + 2\,(v - v_2)$$

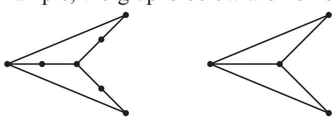
$$\therefore \quad e_1 - v_1 = e_2 - v_2 \quad \text{as required.}$$

5 If G is Eulerian, then all of its vertices have even order.

Now when we add vertices to G and H in order to form homeomorphic graphs, the degrees of the original vertices of G do not change. Furthermore, since we only add vertices of degree 2, then the resulting graph has only vertices of even degree, and hence this graph is Eulerian also.

By the same argument, H must only have vertices of even degree, and H is therefore Eulerian.

6 No. For example, the graphs below are homeomorphic:

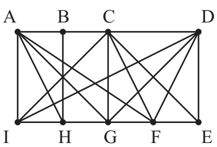


but only the graph on the right is Hamiltonian.

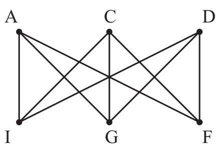
7 Every complete graph K_n where $n > 5$ has K_5 as a subgraph.

$\therefore$ by the theorem of Kuratowski, K_n is non-planar for $n \geqslant 5$.

8 **a**



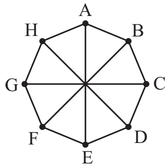
has sub-graph



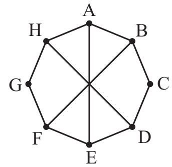
which is $K_{3,3}$.

$\Rightarrow$ the graph is non-planar.

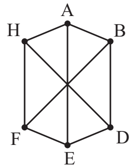
b



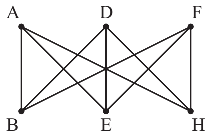
has subgraph



which is homeomorphic to



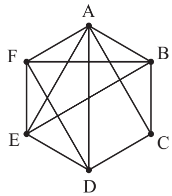
This is isomorphic to



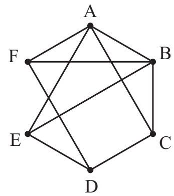
which is $K_{3,3}$.

$\Rightarrow$ the graph is non-planar.

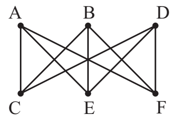
9 **a**



has subgraph



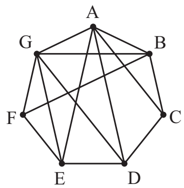
This can be redrawn as



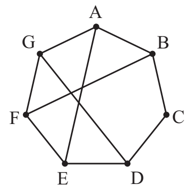
which is $K_{3,3}$.

$\therefore$ by the theorem of Kuratowski, the graph is non-planar.

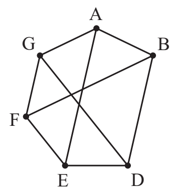
b



has subgraph



which is homeomorphic to



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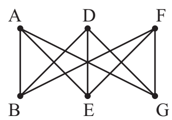
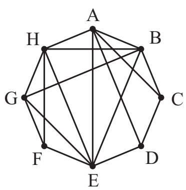


which is $K_{3,3}$.

$\therefore$ by the theorem of Kuratowski, the graph is non-planar.

c No, since the graph is planar.



can be redrawn as