## EXERCISE 11B.6.2 - EXTENSION

**Note:** To make these solutions easier to follow, we have abbreviated "vertex of degree $n$" as $D_n$. So for example, if we refer to $D_2$, it means "vertex of degree 2".

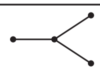
**3 a**

| Degrees | Form | Total Trees |
|---|---|---|
| 1, 1 |  | 1 |

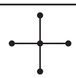
**b**

| Degrees | Form | Permutations | Total Trees |
|---|---|---|---|
| 2, 1, 1 |  | 3 ways to choose $D_2$. | 3 |

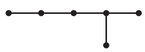
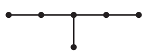
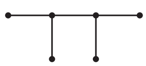
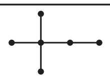
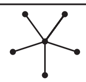
**c**

| Degrees | Form | Permutations | Total Trees |
|---|---|---|---|
| 2, 2, 1, 1 |  | 4! ways to choose the order of vertices,<br>$\div\,2$ because the trees are the same in reverse. | 12 |
| 3, 1, 1, 1 |  | 4 ways to choose $D_3$. | 4 |
| | | | 16 |

**d**

| Degrees | Form | Permutations | Total Trees |
|---|---|---|---|
| 2, 2, 2,<br>1, 1 |  | 5! ways to choose the order of vertices,<br>$\div 2$ because the trees are the same in reverse. | 60 |
| 3, 2, 1,<br>1, 1 |  | 5 ways to choose $D_3$,<br>$\times\,4$ ways to choose $D_2$,<br>$\times\,4$ ways to choose $D_1$ that is adjacent to $D_2$. | 60 |
| 4, 1, 1,<br>1, 1 |  | 5 ways to choose $D_4$. | 5 |
| | | | 125 |

**e**

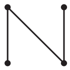
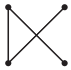
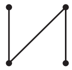
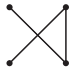
| Degrees | Form | Permutations | Total Trees |
|---|---|---|---|
| 2, 2, 2,<br>2, 1, 1 |  | 6! ways to choose the order of vertices,<br>$\div\,2$ because the trees are the same in reverse. | 360 |
| 3, 2, 2,<br>1, 1, 1 |  | 6! ways to choose the order of vertices,<br>$\div\,2$ since 2 $D_1$ are interchangeable. | 360 |
| 3, 2, 2<br>1, 1, 1 |  | 6! ways to choose the order of vertices,<br>$\div\,2$ since 2 $D_2$ are interchangeable. | 360 |
| 3, 3, 1,<br>1, 1, 1 |  | 6! ways to choose the order of vertices,<br>$\div\,2$ since 2 $D_3$ are interchangeable,<br>$\div\,2$ for the first lot of 2 interchangeable $D_1$,<br>$\div\,2$ for the second lot of 2 interchangeable $D_1$. | 90 |
| 4, 2, 1,<br>1, 1, 1 |  | 6! ways to choose the order of vertices,<br>$\div\,3!$ since 3 $D_1$ are interchangeable. | 120 |
| 5, 1, 1,<br>1, 1, 1 |  | 6 ways to choose $D_5$. | 6 |
| | | | 1296 |

Notice that $1 = 2^0$, $3 = 3^1$, $16 = 4^2$, $125 = 5^3$, $1296 = 6^4$.
Hence for $K_n$, we postulate there are $n^{n-2}$ trees.

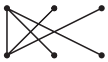
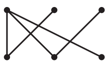
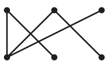
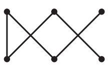
**4 a**

| Degrees | Form | Total Trees |
|---|---|---|
| {1}, {1} |  | 1 |

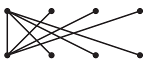
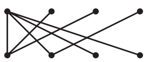
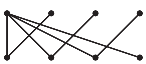
**b**

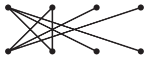
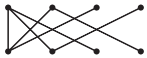
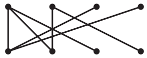
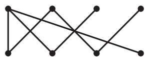
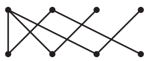
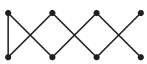
| Degrees | Form | Total Trees |
|---|---|---|
| {2, 1}, {2, 1} |     | 4 |

**c**

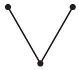
| Degrees | Form | Permutations | Total Trees |
|---|---|---|---|
| {3, 1, 1}, {3, 1, 1} |  | 3 ways to choose $D_3$ on top, $\times$ 3 ways to choose $D_3$ on bottom. | 9 |
| {3, 1, 1}, {2, 2, 1} |  | 3 ways to choose $D_3$ on top, $\times$ 3 ways to choose $D_1$ on bottom, $\times$ 2 ways to choose how $D_2$ on bottom connects to $D_1$ on top. | 18 |
| {2, 2, 1}, {3, 1, 1} |  | symmetric to case {3, 1, 1}, {2, 2, 1} | 18 |
| {2, 2, 1}, {2, 2, 1} |  | 3 ways to choose $D_1$ on top, $\times$ 3 ways to choose $D_1$ on bottom, $\times$ 2 ways to choose which vertex $D_1$ on top connects to, $\times$ 2 ways to choose which vertex $D_1$ on bottom connects to. | 36 |
| | | | 81 |

**d**

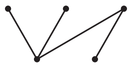
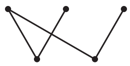
| Degrees | Form | Permutations | Total Trees |
|---|---|---|---|
| {4, 1, 1, 1}, {4, 1, 1, 1} |  | 4 ways to choose $D_4$ on top, $\times$ 4 ways to choose $D_4$ on bottom. | 16 |
| {4, 1, 1, 1}, {3, 2, 1, 1} |  | 4 ways to choose $D_4$ on top, $\times$ 4 ways to choose $D_3$ on bottom, $\times$ 3 ways to choose $D_2$ on bottom, $\times$ 2 ways to choose which $D_1$ the $D_2$ connects to. | 144 |
| {4, 1, 1, 1}, {2, 2, 2, 1} |  | 4 ways to choose $D_4$ on top, $\times$ 4 ways to choose $D_1$ on bottom, $\times$ 3 ways to choose which $D_1$ the first $D_2$ connects to, $\times$ 2 ways to choose which $D_1$ the second $D_2$ connects to. | 96 |
| {3, 2, 1, 1}, {4, 1, 1, 1} | | symmetric to case {4, 1, 1, 1}, {3, 2, 1, 1} | 144 |

| | | | |
|---|---|---|---|
| $\{3, 2, 1, 1\}$,<br>$\{3, 2, 1, 1\}$ | <br><br> | 4 ways to choose $D_3$ on top,<br>$\times\,3$ ways to choose $D_2$ on top,<br>$\times\,4$ ways to choose $D_3$ on bottom,<br>$\times\,3$ ways to choose $D_2$ on bottom,<br>$\times$ (1 way if the $D_3$ aren't adjacent<br>$+\,4$ ways if the $D_3$ are adjacent and both $D_3$ also connect to $D_2$,<br>$+\,4$ ways if the $D_3$ are adjacent and exactly one $D_3$ also connects to $D_2$ (this is 2 for which $D_3$, $\times\,2$ for which $D_1$ the other $D_3$ connects to)) | 1296 |
| $\{3, 2, 1, 1\}$,<br>$\{2, 2, 2, 1\}$ | <br> | 4 ways to choose $D_3$ on top,<br>$\times\,3$ ways to choose $D_2$ on top,<br>$\times\,4$ ways to choose $D_1$ on bottom.<br>So, there are 48 times all of the following:<br>• If $D_3$ is adjacent to $D_1$, we have:<br>3 choices for which $D_2$ the $D_3$ does *not* connect to,<br>$\times\,2$ choices for the connections of the $D_2$ on top (it *must* connect to the $D_2$ on the bottom that the $D_3$ does not connect to, then has 2 options for the other connection),<br>$\times\,2$ choices for which way the $D_1$ are connected.<br>• If $D_3$ is *not* adjacent to $D_1$, we have:<br>3 choices for the connections of the $D_2$ on top (it *must* connect to the $D_1$ on the bottom, then has 3 options for the other connection),<br>$\times\,2$ choices for which way the $D_1$ are connected,<br>i.e., we have $48 \times (12 + 6)$ ways in total. | 864 |
| $\{2, 2, 2, 1\}$,<br>$\{4, 1, 1, 1\}$ | | symmetric to case<br>$\{4, 1, 1, 1\}$, $\{2, 2, 2, 1\}$ | 96 |
| $\{2, 2, 2, 1\}$,<br>$\{3, 2, 1, 1\}$ | | symmetric to case<br>$\{3, 2, 1, 1\}$, $\{2, 2, 2, 1\}$ | 864 |
| $\{2, 2, 2, 1\}$,<br>$\{2, 2, 2, 1\}$ |  | 4 ways to choose $D_1$ on top,<br>$\times\,4$ ways to choose $D_1$ on bottom,<br>$\times\,3$ ways to choose vertex on top (call this $A$) to connect to $D_1$ on bottom,<br>$\times\,3$ ways to choose vertex on bottom (call this $B$) to connect to $D_1$ on top,<br>$\times\,2$ ways to choose other vertex adjacent to $A$,<br>$\times\,2$ ways to choose other vertex adjacent to $B$. | 576 |
| | | | 4096 |

Notice that $1 = 1^0$, $4 = 2^2$, $81 = 3^4$, $4096 = 4^6$.
Hence for $K_n$, we postulate there are $n^{2n-2}$ trees.

**5** For $K_{2,1}$ we have only 1 tree: 

By symmetry, $K_{1,2}$ must only have 1 tree also.
For $K_{3,2}$ we have:

| Form | Permutations | Total Trees |
|---|---|---|
|  | 2 ways to choose $D_3$ on bottom, $\times\,3$ ways to choose which vertex $D_1$ on bottom connects to. | 6 |
|  | 3 ways to choose $D_2$ on top, $\times\,2$ ways to choose which vertex the first $D_1$ on top connects to. | 6 |
| | | 12 |

So, using question **4**, we have these results:

$$\begin{array}{ll} K_{1,1} & 1 = 1^0 \times 1^0 \\ K_{1,2} & 1 = 1^1 \times 2^0 \\ K_{2,1} & 1 = 2^0 \times 1^1 \\ K_{2,2} & 4 = 2^1 \times 2^1 \\ K_{3,2} & 12 = 3^1 \times 2^2 \\ K_{3,3} & 81 = 3^2 \times 3^2 \quad \text{etc.} \end{array}$$

Hence for $K_{m,n}$, we postulate there are $m^{n-1}n^{m-1}$ trees.